Solution by V. M. SPUNAR, Cleveland, Ohio, and the PROPOSER.

The general term is

$$\frac{1}{(4n-3)(4n-2)(4n-1)4n}=\tfrac{1}{6}\left(\frac{1}{4n-3}-\frac{3}{4n-2}+\frac{3}{4n-1}-\frac{1}{4n}\right)$$

$$=\tfrac{1}{6}\left(\frac{2}{4n-3}-\frac{2}{4n-2}+\frac{2}{4n-1}-\frac{2}{4n}\right)-\tfrac{1}{6}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)-\tfrac{1}{6}\left(\frac{1}{4n-2}-\frac{1}{4n}\right).$$

Therefore the series may be written

$$\tfrac{1}{3}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots)-\tfrac{1}{6}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots)-\tfrac{1}{12}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots)$$

$$=\tfrac{1}{4}\log 2-\tfrac{1}{6}\tan^{-1}1=\tfrac{1}{4}\log 2-\tfrac{1}{24}\pi.$$

343. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

A, on contracting to execute a piece of work for \$300 and finding after working alone one day that he had finished but 1% of the entire work, engaged B to assist him at the beginning of the second day, with the understanding, that B on each day was to do 6% as much work as had been completed previously, while A each day was to do an amount of work equal to 1% of the unfinished work at the close of the day before. At the completion of all the work the \$300 were divided between A and B in proportion to the amount of the work each had performed.

Required—(1) The number of days to do the work; (2) on which day would the daily earnings of A and B be the same; and (3) the amount of money each was paid under the agreement.

Solution by the PROPOSER.

Let $r=1\%$ and $r_1=6\%$; and let x, a variable, $=$the time in days to do the whole work, or 1; and let the whole work completed to the end of the days 0, 1, 2, 3, ..., x, $x+1$, ..., be represented by the functions u_0, u_1, u_2, ..., u_x, u_{x+1}, ... The whole work completed to the end of the day $x+1$, or u_{x+1}, is equal to the work completed to the end of x days, or u_x, plus the part completed by A on the day x, or $r(1-u_x)$, plus the part completed by B on the day x, or r_1u_x. Equate the functions and have:

$$u_{x+1}=u_x+r(1-u_x)+r_1u_x\ldots(1);$$
$$\text{or, } u_{x+1}-(1-r+r_1)u_x=r\ldots(2).$$

Give the equation numerical values and we have

$$u_{x+1}-(1.05)u_x=0.01\ldots(3).$$

Equation (3) belongs to the Calculus of Finite Differences. Integrate it and have

$$u_x - C(1.05)^x = -0.2 \ldots (4).$$

Equation (4) is true for all values of x, and is therefore true when $x=0$. When $x=0$, $C=0.2$; and this value of C gives

$$u_x = 0.2[(1.05)^x - 1] \ldots (5).$$

To find (1) the time to complete the work. When the work is completed $u_x=1$. Substitute this value of u_x in (5) and have, $(1.05)^x=6$; or $x\log(1.05)=\log 6$; or $x=\log 6/\log(1.05)=36.72+$days.

To find (2), when the earnings of each are the same, in equation (1) transfer u_x to the first member and have

$$u_{x+1} - u_x = r(1-u_x) + r_1 u_x \ldots (6).$$

As u_{x+1} is the whole work completed in $x+1$ days, and u_x is the whole work completed in x days, their difference is the work completed on the day $x+1$. Substitute in the second member of (6), the value of u_x from (5) and have

$$u_{x+1} - u_x = 0.002[6-(1.05)^x] + 0.012[(1.05)^x - 1] \ldots (7).$$

In (7), for $x+1$ write x, as x is a variable, and have

$$u_x - u_{x-1} = 0.002[6-(1.05)^{x-1}] + 0.012[(1.05)^{x-1} - 1] \ldots (8).$$

The first member of (8) is the work completed in x days, and the two terms of the second member show the work completed by A and B, respectively, on the day x, and as these two terms, under the conditions of the problem, must be equal, equate them, reduce, and have

$$6-(1.05)^{x-1} = 6[(1.05)^{x-1} - 1] \ldots (9);$$

or $7(1.05)^{x-1}=12$; or $(1.05)^{x-1}=12\div 7$; or $(x-1)\log(1.05)=\log(12\div 7)$;
or $x=1+\log(12\div 7)/\log(1.05)=12.05$ days, or 12 days.

To divide the money (3) recur to equation (8) and observe that the exponents of the two terms in the second member are one degree lower than

the subscript x in u_x. This is a general law and it enables us to generate the 36 equations of work completed as x takes different values from 1 to 36.

End of 1 day, $u_1-u_0=0.002[6-(1.05)^1]+[0]\ldots(10)$;

End or 2 days, $u_2-u_1=0.202[6-(1.05)^1]+0.012[(1.05)^1-1]\ldots(11)$;

. . . .

End of 35 days, $u_{35}-u_{34}=0.002[6-(1.05)^{34}]+0.012[(1.05)^{34}-1]\ldots(44)$; and

End of 36 days, $u_{36}-u_{35}=0.002[6-(1.05)^{35}]+0.012[(1.05)^{35}-1]\ldots(45)$.

Add the equations and have

$$u_{36}-u_0=0.002\{216-[(1.05)^0+(1.05)^1+\ldots+(1.05)^{35}]\}$$
$$+0.012[(1.05)^1+(1.05)^2+\ldots(1.05)^{35}-35]\ldots(46).$$

Sum the first term of the second member of (46) for the work completed by A in 36 days, and sum the second term for the work completed by B in 35 days, and have:

$$\text{for A's work, } 0.002\{216-20[(1.05)^{36}-1]\}=0.2403+;$$
$$\text{and for B's work, } 0.012\{21[(1.05)^{35}-1]-35\}=0.7180+.$$

A's work for 36 days+B's work for 35 days$=0.2403+0.7180=0.9583+$.

The unfinished work$=1-0.9583=0.0417-$; work to be finished by A and B in 0.72 day.

For A's unfinished part we have $(0.0417)(0.01)(0.72)=0.0003$; and $0.2403+0.0003=0.2406=$the total part completed by A.

For B's unfinished part we have $(0.8583)(0.06)(0.72)=0.0414$; and $0.7180+0.0414=0.7594$, the total part completed by B.

A's total+B's total$=0.2406+0.7594=1$, as it should.

A's share of the money, therefore, $=\$300\times(0.2406)=\$72.18$; and B's share$=\$300\times(0.7594)=\$227.82$.

Also solved by V. M. Spunar.

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

No satisfactory solution of this problem has been received.